5.5 Completeness for First-order Logic

Definition 5.14 *$\{(\exists a)\beta \supset \beta(b/a) | $ for all $a, b \in \mathsf{L}_{\mathsf{NC}^1}\}$ is the* **E-form** *on β.*

Definition 5.15 *Γ is* **fully E-formed** *if for every $\beta \in \mathsf{L}_{\mathsf{NC}^1}$, $(\exists a)\beta \supset \beta(b/a) \in \Gamma$ for some b not free in β.*

Definition 5.16 *Γ is* **fully witnessed** *if whenever $(\exists a)\beta \in \Gamma$ then $\beta(b/a) \in \Gamma$ for some b not free in β (i.e., such that either b does not occur in β, or if it does, its only occurrences are bound). We can think of b as a* **witness** *to the truth of $(\exists a)\beta$.*

5.5.1 The Way Up

The work of this section is purely syntactic—i.e., proof theory. The idea is to take any simply consistent set Γ, and extend it to a maximally consistent fully witnessed set Δ. (Δ is in a sense a purely syntactic *Ersatz* (substitute) for a model.)

Exercise 5.11 *Check that Lemmas 5.1 and 5.2 can be extended to NC^1.*

Lemma 5.6 *Every fully E-formed set maximally consistent in $\mathsf{L}_{\mathsf{NC}^1}$ is fully witnessed.*

Exercise 5.12 *Prove Lemma 5.6.*

Lemma 5.7 *Every set Γ simply consistent in $\mathsf{L}_{\mathsf{NC}^1}$ can be extended to a fully E-formed one.*

Proof: Let $\{\Pi_n | n \in \omega\}$ be an enumeration of the E-forms.
Define $\Gamma^0 = \Gamma$.
$\Gamma^{n+1} = \Gamma^n \cup \{\alpha_n\}$ for some $\alpha_n \in \Pi_n$ such that, where $\alpha_n = (\exists a)\beta \supset \beta(b/a)$, b is not free in any wff in Γ^n, nor in β. (It may be necessary in this construction to augment the language with at most countably many new variables.)
Let $\Gamma^\infty = \bigcup_{n\in\omega} \Gamma^n$. Then clearly Γ^∞ is fully E-formed.
We must show Γ^∞ is simply consistent in $\mathsf{L}_{\mathsf{NC}^1}$.
Suppose it is not, i.e., $\Gamma^\infty \vdash \bot$. Then, since derivations are finite, for some n, $\Gamma^n \nvdash \bot$ and $\Gamma^{n+1} \vdash \bot$, i.e., $\Gamma^n \cup \{\alpha_n\} \vdash \bot$.
But $\vdash_{\mathsf{NC}} (\exists b)((\exists a)\beta \supset \beta(b/a))$ since b is not free in $(\exists a)\beta$. (See Exercise 5.13.)
So by $\exists E$, $\Gamma^n \vdash \bot$. But we supposed it was simply consistent.
So Γ^∞ must be simply consistent, as required. Q.E.D.

Exercise 5.13
1. *Show that $\vdash_{\mathsf{NC}} (\exists b)((\exists a)\beta \supset \beta(b/a))$ provided b is not free in $(\exists a)\beta$. (Hint: first show, e.g., $\vdash_{\mathsf{NC}} (\exists y)((\exists x)Fx \supset Fy)$, and then generalize the proof using metavariables a, b and β.)*
2. *Explain and justify the restriction, 'provided b is not free in $(\exists a)\beta$'.*

Lemma 5.8 *Every simply consistent set can be extended to a maximally consistent fully witnessed set.*

Proof: Let Γ be simply consistent. Extend Γ to Γ^∞, fully E-formed by Lemma 5.7. Extend Γ^∞ to Δ, maximally consistent by Exercise 5.11. Clearly the procedure retains E-formedness. Then Δ is fully witnessed, by Lemma 5.6. Q.E.D.

5.5.2 The Way Down

Now we start the semantics. We must find a model for Δ—which will be a model for Γ, since $\Gamma \subseteq \Delta$.

Lemma 5.9 *Every maximally consistent fully witnessed set* Δ *has a characteristic model.*

Proof: We show Δ has a model by defining its **canonical** model. Let v be an interpretation such that:

$$\begin{aligned} D_v &= \{a | a \text{ is a variable in some wff of } \Delta\}. \\ v(a) &= a, \text{ for all variables } a. \\ v(\phi^n) &= \{\langle a_1, \dots a_n\rangle | \phi a_1 \dots a_n \in \Delta\}. \end{aligned}$$

Then v is a characteristic model for Δ. We show this by induction on the degree of α.

Base deg $(\alpha) = 0$, so $\alpha = \phi^n a_1 \dots a_n$

Then α is true under v iff $v(\alpha) = 1$
iff $\langle v(a_1), \dots v(a_n)\rangle \in v(\phi)$
iff $\langle a_1, \dots a_n\rangle \in \{\langle a_1, \dots a_n\rangle | \phi a_1 \dots a_n \in \Delta\}$
iff $\phi a_1 \dots a_n \in \Delta$ iff $\alpha \in \Delta$ as required.

Induction step

1. $\alpha = \bot$. Then $v(\alpha) \neq 1$ and $\bot \notin \Delta$ (since Δ is simply consistent)
so vacuously $v(\alpha) = 1$ iff $\alpha \in \Delta$.
2. $\alpha = \beta \wedge \gamma$. The I.H. says that β is true under v iff $\beta \in \Delta$ and that γ is true under v iff $\gamma \in \Delta$.
So α is true under v iff $v(\alpha) = 1$ iff $v(\beta) = v(\gamma) = 1$ iff $\beta \in \Delta$ and $\gamma \in \Delta$ iff $\beta \wedge \gamma \in \Delta$ by Lemma 5.1(3), i.e., $\alpha \in \Delta$.
3. $\alpha = \beta \vee \gamma$. See Exercise 5.14.
4. $\alpha = \beta \supset \gamma$. See Exercise 5.14.
5. $\alpha = (\forall a)\beta$. To show $v(\alpha) = 1$ iff $\alpha \in \Delta$.
First, suppose $\alpha \notin \Delta$. Then $\neg\alpha \in \Delta$ (for Δ is maximally consistent)
so $\Delta \vdash \neg(\forall a)\beta$, so $\Delta \vdash (\exists a)\neg\beta$, whence $(\exists a)\neg\beta \in \Delta$ by Lemma 5.1(1)
so $\neg\beta(b/a) \in \Delta$ for some b not free in β, since Δ is fully witnessed,
so $\beta(b/a) \notin \Delta$.

I.H. : $v(\beta(b/a)) = 1$ iff $\beta(b/a) \in \Delta$. So $v(\beta(b/a)) = 0$.
Consider $v' \underset{a}{\sim} v$ with $v'(a) = b = v(b)$.
Then by Lemma 5.5, $v'(\beta) = 0$, i.e., β is not true under v'. Thus $(\forall a)\beta$ is not true under v.
Contraposing, if α is true under v then $\alpha \in \Delta$.
Conversely, suppose $\alpha \in \Delta$. Consider any $v' \underset{a}{\sim} v$, $v'(a) = b$, say.

(a) if b is free for a in β, then, since $\Delta \vdash \alpha$, $\Delta \vdash \beta(b/a)$ by $\forall$E. So $\beta(b/a) \in \Delta$.
I.H. $v(\beta(b/a)) = 1$ iff $\beta(b/a) \in \Delta$. So $\beta(b/a)$ is true under v. But $v' \underset{a}{\sim} v$ and $v'(a) = b$. So by Lemma 5.5, β is true under v'. So by the truth definition, $\alpha = (\forall a)\beta$ is true under v.

(b) However, if b is not free for a in β, then we form a bound alphabetic variant β' of β by substituting c, not occurring in β, for every bound occurrence of b in β. Then by Exercise 5.10, for all u, β' is true under u iff β is true under u. Moreover $\vdash_{\mathsf{NC}} \beta \equiv \beta'$. (See Exercise 5.15) So, since $(\forall a)\beta \in \Delta$, $(\forall a)\beta' \in \Delta$ by deductive closure. Hence, as in (a), $(\forall a)\beta'$ is true under v, and so $(\forall a)\beta$ is true under v.

6. $\alpha = (\exists a)\beta$. See Exercise 5.14. Q.E.D.

Exercise 5.14 *Prove parts 3, 4 and 6 in Lemma 5.9.*

Exercise 5.15 *1. Give a formal definition of '(bound) alphabetic variant'.*

2. Let α' be a bound alphabetic variant of α. Show that $\vdash_{\mathsf{NC}} \alpha \equiv \alpha'$. (Hint: use induction.)

Lemma 5.10 *(The Existence Lemma for NC^1) Every simply consistent set has a model.*

Proof: By Lemmas 5.8 and 5.9. Q.E.D.

5.5.3 Putting it All Together

Theorem 5.3 *(Henkin Completeness theorem for NC^1) If $\Gamma \models \alpha$ then $\Gamma \vdash_{\mathsf{NC}} \alpha$.*

Proof: Suppose $\Gamma \models \alpha$. Then there is no model of $\Gamma \cup \{\neg\alpha\}$. So by Lemma 5.10, $\Gamma \cup \{\neg\alpha\}$ is not simply consistent (in NC). Hence by Exercise 5.5, $\Gamma \vdash_{\mathsf{NC}} \alpha$. Q.E.D.

Further Reading

J. Zalabardo, *Introduction to the Theory of Logic*, ch. 5.
B. Mates, *Elementary Logic*, 136-42.
R. Stoll, *Set Theory and Logic* § 9.6.
W.V. Quine, *Mathematical Logic* §21 'Alphabetic Variance', pp. 109-15.
L. Henkin, 'The Completeness of the First-order Functional Calculus', *Journal of Symbolic Logic* 14, 1949, 159-66.

5.6 Soundness for Predicate Logic

Theorem 5.4 *If* $\Gamma \vdash_{\mathsf{NC}} \alpha$ *then* $\Gamma \models \alpha$*.*

Proof: By induction on length of proof, as before. There are four more cases to consider, beyond those in § 5.2:

$\exists$**I:** $\langle \Gamma, (\exists a)\alpha \rangle$ is an i.c. of $\langle \Gamma, \alpha(b/a) \rangle$.

So the proof of $\langle \Gamma, \alpha(b/a) \rangle$ is shorter, whence by I.H., $\Gamma \models \alpha(b/a)$.

To show $\Gamma \models (\exists a)\alpha$. Let v model Γ.

Then $\alpha(b/a)$ is true under v.

Consider $v' \underset{a}{\sim} v$ with $v'(a) = v(b)$.

Then by Lemma 5.5, α is true under v', i.e. $v'(\alpha) = 1$ for some $v' \underset{a}{\sim} v$.

Hence by the truth definition, $v((\exists a)\alpha) = 1$.

Thus $\Gamma \models (\exists a)\alpha$ as required.

$\forall$**E:** see Exercise 5.16.

$\forall$**I:** $\langle \Gamma, (\forall a)\alpha \rangle$ is an i.c. of $\langle \Gamma, \alpha(b/a) \rangle$ where b is free in no member of Γ, nor in $(\forall a)\alpha$.

Clearly the derivation of $\langle \Gamma, \alpha(b/a) \rangle$ is shorter, so by I.H., $\Gamma \models \alpha(b/a)$. We must show $\Gamma \models (\forall a)\alpha$. Let v be a model of Γ. Take $v' \underset{a}{\sim} v$. Suppose $v'(a) = e$. Then consider $v'' \underset{b}{\sim} v$, with $v''(b) = e$. Since b is free in no member of Γ, v'' is a model of Γ, by Exercise 5.10. So by I.H., $\alpha(b/a)$ is true under v''. Let $v''' \underset{b}{\sim} v'$ and $v''' \underset{a}{\sim} v''$. Then $v'''(a) = e$, and $v'''(b) = e$.

$$\begin{array}{ccc} v & \underset{a}{\sim} & v' \\ \underset{b}{\sim} & & \underset{b}{\sim} \\ v'' & \underset{a}{\sim} & v''' \end{array}$$

Then, since $\alpha(b/a)$ is true under v'', and since a is not free in $\alpha(b/a)$, $\alpha(b/a)$ is true under v''', by Exercise 5.10.

We now apply Lemma 5.5 to v''' and v', given that $v' \underset{b}{\sim} v'''$ and $v'''(b) = v'(a)$, to conclude that $\alpha(b/a)(a/b)$ is true under v'.

But since b is not free in $(\forall a)\alpha$, $\alpha(b/a)(a/b) = \alpha$.

So α is true under v', where v' is any a-variant of v.

Hence $(\forall a)\alpha$ is true under v, as required.

$\exists$**E:** see Exercise 5.16. Q.E.D.

Exercise 5.16 *Complete the proof of Theorem 5.4 by covering the cases of* $\forall E$ *(like* $\exists I$*) and* $\exists E$ *(like* $\forall I$*).*

Exercise 5.17 *Show by examples that without the restrictions on the free variable ('b') in* $\exists E$*, invalid sequents can be proved. Give examples where 'b' is free, first, in* β*, then in* $(\exists a)\alpha$ *(cf. Exercise 5.13 (2)) and finally in* $\Delta - \{\alpha(b/a)\}$*.*